Feedback Based Dynamic Congestion Pricing

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Abstract

This paper presents a mathematical model for dynamic congestion pricing at a toll where alternate lane or routes are available. The model developed is based on traffic conservation law and queueing, and moreover uses fundamental macroscopic relationships for its derivation. The modeling uses a Logit model for the price and driver choice behavior relationship. We use this nominal mathematical model for the dynamics to derive a feedback control law that uses real-time information to come up with tolling price. Simulation results show the performance of this dynamic feedback congestion pricing algorithm.

INTRODUCTION

Congestion pricing is defined as charging motorists during peak hours to encourage them to either switch their travel times or to use alternative routes which are not congested at peak hours. The theory behind road pricing suggests that, in order to reach social optimum, a toll needs to be charged which is equal to the difference between social marginal costs (which include external costs that users impose on each other on a congested road) and private average costs of users(travel delays, fuel, maintenance etc.).

In recent years, with the help of technological developments such as electronic toll collection system, pricing can also be done dynamically, that is, tolls can be set real-time varied according to the traffic conditions. Although the continuously time-varying optimal tolls suggest a fair system for the users, it is also debatable whether smoothly-varying toll rate will be appreciated by drivers. Therefore, in real world dynamic pricing applications step (piecewise constant) tolls are mainly used. Examples from US are depicted in Table 1.

TABLE 1 Dynamic Pricing Applications in US

FACILITY	TOLLING SYSTEM	WEBSITE
San Diego I-15 FasTrak	The toll schedule varies dynamically every 6 minutes ^{footnote}	http://argo.sandag.org/fastrak
Orange County, CA SR-91	Toll varies every hour depending on traffic conditions	www.91expresslanes.com
Minnesota I-394	Tolls can be varied as frequently as 3 minutes ^{footnote}	http://www.mnpass.org/394/index.html

BACKGROUND AND MOTIVATION

As mentioned in the previous section, dynamic congestion pricing application currently make use of step functions that dynamically adjust toll rates based on the prevailing traffic conditions on the toll road. Clearly, this approach has several shortcomings including the lack of theoretical basis for the determination and implementation of tolls. Moreover, such an approach can cause unpredicted fluctuations in travel times and overall sub-optimal results in terms of users as well as the system.

In this paper, we will propose a theoretically sound feedback based congestion pricing model that will be attempt to:

1. Achieve the pre-set objective such as system optimal or allowable user-equilibrium.
2. Develop a control law that is ribust against uncertainties within a set.
3. Assure that the developed control law is stable and does not fluctuate in an implementable way. For example, if the dynamic toll prices changes from $5 to $20 and then to $5 in a very short period of time, say 5 minutes, the, this will create unexpected results and low compliance rates. Thus, dynamic toll prices should increase and decrease in a relatively smooth way.
4. Incorporate bounds for maximum and minimum tolls to ensure equity.

LITERATURE REVIEW

Congestion pricing has been one of the most important research topics in traffic engineering throughout the last few decades. Several studies were conducted on the theoretical aspects of pricing models ((*13*), (*5*), (*3*), Byung-Wook Wie and Tobin, 1998, (*2*)). Lindsey (2003) reviewed road pricing applications in the US

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[1] http://ops.fhwa.dot.gov/tolling_pricing/value_pricing/projects/involving_tolls /priced_lanes/hot_lanes/ca_hotlanes_i15sd.hm

[2] http://www.tollroadsnews.com/node/1197

and Canada by comparing the implementations in Europe and applicability of the different categories of congestion pricing to US roads (*11*). Other international congestion pricing experiences such as Singapore, Norway and United Kingdom were explained and lessons learned from these implementations were also analyzed by different authors ((*10*), (*7*), (*12*)). In practice, congestion pricing is performed generally by 1) HOT (High-occupany toll lanes) lanes, which are the lanes reserved for vehicles that meet minimum occupant requirements or vehicles that pay tolls, 2) Cordon pricing, which is charging vehicles to access a zone (e.g. highly congested part of a metropolitan city). Some of the major road pricing applications in the US are summarized in Table 2

TABLE 2 Major Road Pricing Initiatives in the US

(a) Dynamic Pricing Applications

Project/Location	**Initiation Date**	**For More Information**
I-15 San Diego, CA	1998	*http : //www.sandag.org/index.asp?projectid = 34fuse action = projects.detail.*
I-394 Minneapolis,MN	2005	*http : //www.mnpass.org/phase2.html*
SR 167 Puget Sound Region,WA	2008	*http : //www.wsdot.wa.gov/Projects/SR167/HOTLanes/*
SR 91 Orange County, CA	1995	*http : //ceenve.calpoly.edu/sullivan/sr91/sr91.htm*
I-95 Miami,FL	2008	*http : //ops.fhwa.dot.gov/tolling_pricing/value_pricing/ projects/involving_tolls/priced_lanes/hot_lanes/fl_ hotlanes_i95miami.htm*

(b) Time-of-Day Pricing Applications

Project/Location	**Initiation Date**	**For More Information**
New Jersey Turnpike Variable Tolls, NJ	2000	*http : //knowledge.fhwa.dot.gov/cops/hcx.nsf/384 aefcefc48229e85256a71004b24e0/ba2414ce1eac182685 256dc500674090?OpenDocument.*
Variable Pricing of Bridges, Lee Co., FL	1998	*http : //leewayinfo.com/*
Variable Tolls on N.Y. Hudson RiverCrossings	2001	*http : //www.panynj.gov/tbt/tbtframe.HTM*
I-25/US 36 Denver	2006	*http : //www.its.dot.gov/jpodocs/repts_te/13668_ files/chapter_7.htm*

Although congestion pricing is generally studied for simple settings such as static networks and homogeneous users, there are also several studies concerning real world conditions. Holguin-Veras and Cetin (2009) studied optimal tolls for multi-class traffic and generated analytical formulations (*6*), De Palma et al. (2005) analyzed time varying tolls considering departure time, route choice, mode split in a dynamic network equilibrium (*1*), Chen and Berstein (2004) conducted a study tolling for different user types (*9*). Ozbay and Yanmaz-Tuzel (2008) conducted a study on the valuation of travel time and departure time choice under congestion pricing, considering the New Jersey Turnpike's value pricing implementation (*8*). There have also been some recent attempts for developing real-time dynamic congestion pricing algorithms. Zhang et al. (2008) created a feedback-based tolling algorithm for high-occupancy toll lane facilities. In their algorithm, the feedback control is obtained by a step-wise function monitoring the speeds of general purpose lanes and HOT lanes and toll rates are estimated by backward calculation using logit model. Simulation results of the model showed that overall traffic conditions were improved significantly (*4*).

SYSTEM MODEL

Most of the real-world dynamic toll pricing projects in the US is based on the existence of a toll road and a tool free road as an alternative. Commuters are this expected to make a decision about choosing the toll road at a decision point where the prevailing traffic conditions on both roadways as well as the current tolls are communicated to them mainly through variable message sigs. Thus, each traveler makes a decision as to whether or not to pay the toll and use the toll road or to simply continue to drive on the free alternative.

A modified version of feedback routing model developed by Kachroo and Ozbay (2005, 1998a, b) with some modifications to its route cost functions, can be used as a mechanism to regulate traffic coming to the single decision model. However, it is important to first analyze that model in terms of the above requirements specific to the congestion pricing problem.

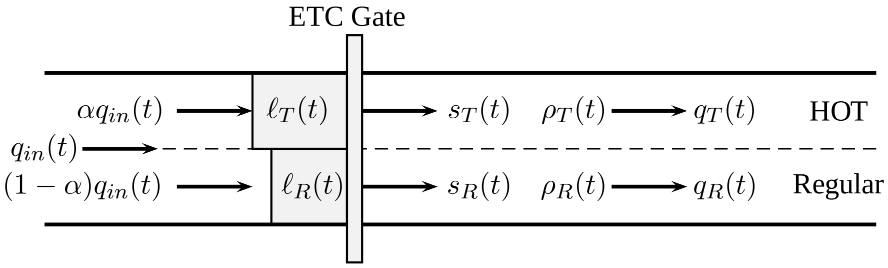

FIGURE 1 System Configuration

The system dynamics are given by:

$$\begin{aligned}
\dot{\ell_T} &= \alpha q_{in}(t) - s_T(t) \\
\dot{\rho_T} &= s_T(t) - v_f \rho_T \left(1 - \frac{\rho_T(t)}{\rho_m}\right) \\
\dot{\ell_R} &= (1-\alpha) q_{in}(t) - s_R(t) \\
\dot{\rho_R} &= s_R(t) - v_f \rho_R \left(1 - \frac{\rho_R(t)}{\rho_m}\right)
\end{aligned} \tag{1}$$

The symbols for different variables are shown in Table 3.

TABLE 3 Symbols used in the Mathematical Formulation

SYMBOL	MEANING
q_{in}	Traffic in-flow
α	Percent flow using toll
ℓ_T	Queue length for toll lane
ℓ_R	Queue length for regular lane
s_T	Service rate for toll lane
s_R	Service rate for regular lane
ρ_T	Traffic density in toll lane
ρ_R	Traffic density in regular lane
q_T	Traffic outflow from toll lane
q_R	Traffic outflow from regular lane
L_T	Length of the toll lane
L_R	Length of the regular lane
β	Fraction of toll in-flow using RF-tags
T_T	Travel time through toll lane
T_R	Travel time through regular lane

Using Greenshield's fundamental relationship, which we have already used to derive equation 1, we have:

$$
\begin{aligned}
q_T(t) &= v_f \rho_T \left(1 - \frac{\rho_T(t)}{\rho_m}\right) \\
q_R(t) &= v_f \rho_R \left(1 - \frac{\rho_R(t)}{\rho_m}\right)
\end{aligned}
\tag{2}
$$

There are various modifications of the basic model in equation 1 that we can use based on the actual implementation of the tolling scheme. For instance if the tolling is done automatically for everyone using RF-tags, then there will be no queues in the system and there will be no ETC gate. Hence, the dynamics for that implementation will have equations that are shown in equation 3

$$
\begin{aligned}
\dot{\rho_T} &= \alpha q_{in}(t) - v_f \rho_T \left(1 - \frac{\rho_T(t)}{\rho_m}\right) \\
\dot{\rho_R} &= (1-\alpha) q_{in}(t) - v_f \rho_R \left(1 - \frac{\rho_R(t)}{\rho_m}\right)
\end{aligned}
\tag{3}
$$

If only the toll lane has a gate and no gate for regular lane, then the model has equations that are shown in equation 4

$$
\begin{aligned}
\dot{\ell_T} &= \alpha q_{in}(t) - s_T(t) \\
\dot{\rho_T} &= s_T(t) - v_f \rho_T \left(1 - \frac{\rho_T(t)}{\rho_m}\right) \\
\dot{\rho_R} &= (1-\alpha) q_{in}(t) - v_f \rho_R \left(1 - \frac{\rho_R(t)}{\rho_m}\right)
\end{aligned}
\tag{4}
$$

If β is the fraction of toll vehicles that use the RF-tages as depicted in Figure 2, then the dynamics will be given by equation 5

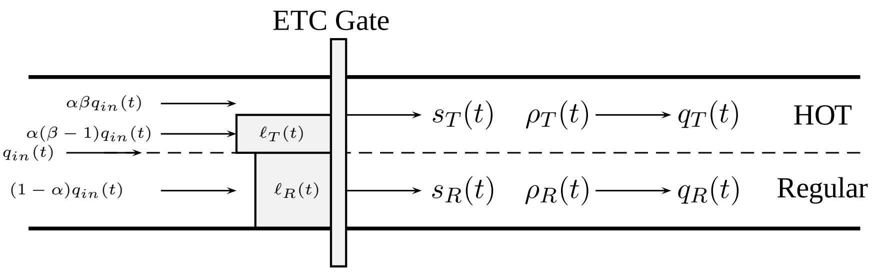

FIGURE 2 System Configuration

$$
\begin{aligned}
\dot{\ell_T} &= \alpha(1-\beta) q_{in}(t) - s_T(t) \\
\dot{\rho_T} &= s_T(t) + \alpha\beta q_{in}(t) - v_f \rho_T \left(1 - \frac{\rho_T(t)}{\rho_m}\right) \\
\dot{\ell_R} &= (1-\alpha) q_{in}(t) - s_R(t) \\
\dot{\rho_R} &= s_R(t) - v_f \rho_R \left(1 - \frac{\rho_R(t)}{\rho_m}\right)
\end{aligned}
\tag{5}
$$

The most general model in our setting would allow queueing in every lane and would have the structure shown in equation 6. In this model, the queue length for tagged vehicles is given by ℓ_{RF} and the service rate as $s_{RF}(t)$.

$$\begin{aligned}
\dot{\ell_T} &= \alpha(1-\beta)q_{in}(t) - s_T(t)\\
\dot{\ell_{RF}} &= \alpha\beta q_{in}(t) - s_{RF}(t)\\
\dot{\rho_T} &= s_T(t) + s_{RF}(t) - v_f\rho_T\left(1 - \frac{\rho_T(t)}{\rho_m}\right)\\
\dot{\ell_R} &= (1-\alpha)q_{in}(t) - s_R(t)\\
\dot{\rho_R} &= s_R(t) - v_f\rho_R\left(1 - \frac{\rho_R(t)}{\rho_m}\right)
\end{aligned} \tag{6}$$

In this model, however, if the lanes are wide enough so that no queues are formed for the tagged and regular vehicles, then we will obtain the model given by equation 7. This is the model we have used in the simulation studies presented in this paper.

$$\begin{aligned}
\dot{\ell_T} &= \alpha(1-\beta)q_{in}(t) - s_T(t)\\
\dot{\rho_T} &= s_T(t) + \alpha\beta q_{in}(t) - v_f\rho_T\left(1 - \frac{\rho_T(t)}{\rho_m}\right)\\
\dot{\rho_R} &= (1-\alpha)q_{in}(t) - v_f\rho_R\left(1 - \frac{\rho_R(t)}{\rho_m}\right)
\end{aligned} \tag{7}$$

FEEDBACK CONTROL LAW FOR THE ROBUST CONGESTION PRICING

In order to derive a feedback control law for allowable user-equilibrium, we will use the formula 8 for travel time through tolled and regular lanes. Drivers would not pay for a facility that would give the same performance as a free facility. Therefore, the travel times of the two facilities can not be the same. However, we can obtain an "allowable" user-equilibrium by maintaining some scaled version of travel time in the tolled lane equal to the regular lane. We use the symbol γ for that factor.

$$\begin{aligned}
T_T(t) &= \frac{\ell_T(t)}{s_T(t)} + \frac{L_T}{v_f\left(1 - \frac{\rho_T(t)}{\rho_m}\right)}\\
T_R(t) &= \frac{\ell_R(t)}{s_R(t)} + \frac{L_R}{v_f\left(1 - \frac{\rho_R(t)}{\rho_m}\right)}
\end{aligned} \tag{8}$$

We would like to make the error defined by equation 9 to have closed loop dynamics that is asymptotically stable in the sense of Lyapunov.

$$\begin{aligned}
&e(t) = \gamma T_T(t) - T_R(t) =\\
&\gamma\left(\frac{\ell_T(t)}{s_T(t)} + \frac{L_T}{v_f\left(1 - \frac{\rho_T(t)}{\rho_m}\right)}\right) - \left(\frac{\ell_R(t)}{s_R(t)} + \frac{L_R}{v_f\left(1 - \frac{\rho_R(t)}{\rho_m}\right)}\right)
\end{aligned} \tag{9}$$

We will use feedback linearization technique to derive the feedback control law. For that design we need to differentiate the error term with respect to time. Hence, differentiating equation 9 gives us the dynamics 10

$$
\begin{aligned}
\dot{e}(t) &= \gamma \dot{T}_T(t) - \dot{T}_R(t) \\
&= \gamma \left(-\frac{\ell_T(t)}{s_T^2(t)}\dot{s}_T(t) + \frac{\dot{\ell}_T(t)}{s_T(t)} + \frac{L_T}{v_f\rho_m\left(1-\frac{\rho_T(t)}{\rho_m}\right)^2}\dot{\rho}_T(t) \right) \\
&- \left(-\frac{\ell_R(t)}{s_R^2(t)}\dot{s}_R(t) + \frac{\dot{\ell}_R(t)}{s_R(t)} + \frac{L_R}{v_f\rho_m\left(1-\frac{\rho_R(t)}{\rho_m}\right)^2}\dot{\rho}_R(t) \right)
\end{aligned}
\tag{10}
$$

Expanding just one term in these error dynamics using system dynamics 5, we get

$$
\begin{aligned}
\dot{T}_T(t) &= -\frac{\ell_T(t)}{s_T^2(t)}\dot{s}_T(t) + \frac{\dot{\ell}_T(t)}{s_T(t)} + \frac{L_T}{v_f\rho_m\left(1-\frac{\rho_T(t)}{\rho_m}\right)^2}\dot{\rho}_T(t) \\
&= -\frac{\ell_T(t)}{s_T^2(t)}\dot{s}_T(t) + \frac{1}{s_T(t)}\left[\alpha(1-\beta)q_{in}(t) - s_T(t)\right] \\
&+ \frac{L_T}{v_f\rho_m\left(1-\frac{\rho_T(t)}{\rho_m}\right)^2}\left[s_T(t) + \alpha\beta q_{in}(t) - v_f\rho_T(t)\left(1-\frac{\rho_T(t)}{\rho_m}\right)\right]
\end{aligned}
\tag{11}
$$

Similarly,

$$
\begin{aligned}
\dot{T}_R(t) &= -\frac{\ell_R(t)}{s_R^2(t)}\dot{s}_R(t) + \frac{\dot{\ell}_R(t)}{s_R(t)} + \frac{L_R}{v_f\rho_m\left(1-\frac{\rho_R(t)}{\rho_m}\right)^2}\dot{\rho}_R(t) \\
&= -\frac{\ell_R(t)}{s_R^2(t)}\dot{s}_R(t) + \frac{1}{s_R(t)}\left[(1-\alpha)q_{in}(t) - s_R(t)\right] \\
&+ \frac{L_R}{v_f\rho_m\left(1-\frac{\rho_R(t)}{\rho_m}\right)^2}\left[s_R(t) - v_f\rho_R(t)\left(1-\frac{\rho_R(t)}{\rho_m}\right)\right]
\end{aligned}
\tag{12}
$$

Substituting equations 11 and 14 into equation 10 gives us

$$
\dot{e}(t) = f + g\alpha \tag{13}
$$

where f and g are state dependent terms whose exact formula can be extracted from using 11 and 14 with equation 10.

Now, we can design the feedback control law for α as

$$
\alpha = g^{-1}\left(-f - ke(t)\right) \tag{14}
$$

Using this control law in dynamic equation 14 shows the asymptotic stability of the error as:

$$
\lim_{t\to\infty} e(t) = 0 \tag{15}
$$

Although we have obtained the closed-loop desired behavior, we still have to come up with the actual toll rate that we must charge. To come up with the functional form for that, we choose a Logit model to formulate the driver decision to choose between the tolled and regular lanes. We use the following relationship:

$$\alpha = \frac{1}{1+\exp\left(a_1(T_T(t)-T_R(t))+a_2p(t)+a_3\right)} \tag{16}$$

Here, $p(t)$ is the toll rate, a_1 is the marginal effect factor of the travel time difference to the driver's utility, a_2 is the marginal effect factor of the toll rate to the same utility, and finally, a_3 covers other factors in the driver choice. From equation 16, we can obtain the deployable toll rate in terms of computed α as

$$p(t) = \frac{1}{a_2}\left(\ln(\alpha-1) - a_1(T_T(t)-T_R(t)) - a_3\right) \tag{17}$$

SIMULATION BASED EVALUATION OF ROBUST CONGESTION PRICING

We use Scilab software to perform simulations for the dynamics for the system given by equation 7. We use the control law given by equation 14. Now, since we are using no queues for the tagged and regular lanes, there will be no terms for the queue lengths in the controller. Moreover, for the sake of simulation we will assume that the service rate for tolling is fixed. Based on these conditions we get

$$f = \gamma\left(\frac{L_T}{v_f\rho_m\left(1-\dfrac{\rho_T(t)}{\rho_m}\right)^2}\left[s_T - v_f\rho_T(t)\left(1-\frac{\rho_T(t)}{\rho_m}\right)\right]-1\right) - \frac{L_R\,\rho_R(t)}{\rho_m\left(1-\dfrac{\rho_R(t)}{\rho_m}\right)} \tag{18}$$

$$g = \gamma\left(\frac{1}{s_T}\left[(1-\beta)q_{in}(t)\right] + \frac{L_T}{v_f\rho_m\left(1-\dfrac{\rho_T(t)}{\rho_m}\right)^2}\left[\beta q_{in}(t)\right]\right) \tag{19}$$

The control law for the simulation is

$$\alpha = g^{-1}\left(-f - ke(t)\right) \tag{20}$$

where

$$e(t) = \gamma\left(\frac{\ell_T(t)}{s_T} + \frac{L_T}{v_f\left(1-\dfrac{\rho_T(t)}{\rho_m}\right)}\right) - \left(\frac{L_R}{v_f\left(1-\dfrac{\rho_R(t)}{\rho_m}\right)}\right) \tag{21}$$

The simulation parameters are given in Table 4.

We use a variable inflow rate to see how the system would evolve. The feedback control law tries to keep the error rate low. In the simulation we also make sure that the implemented value of the split is between zero and one, and also that the queue length and all other state variables always remain non-negative. The simulation results are shown in Figure 3.

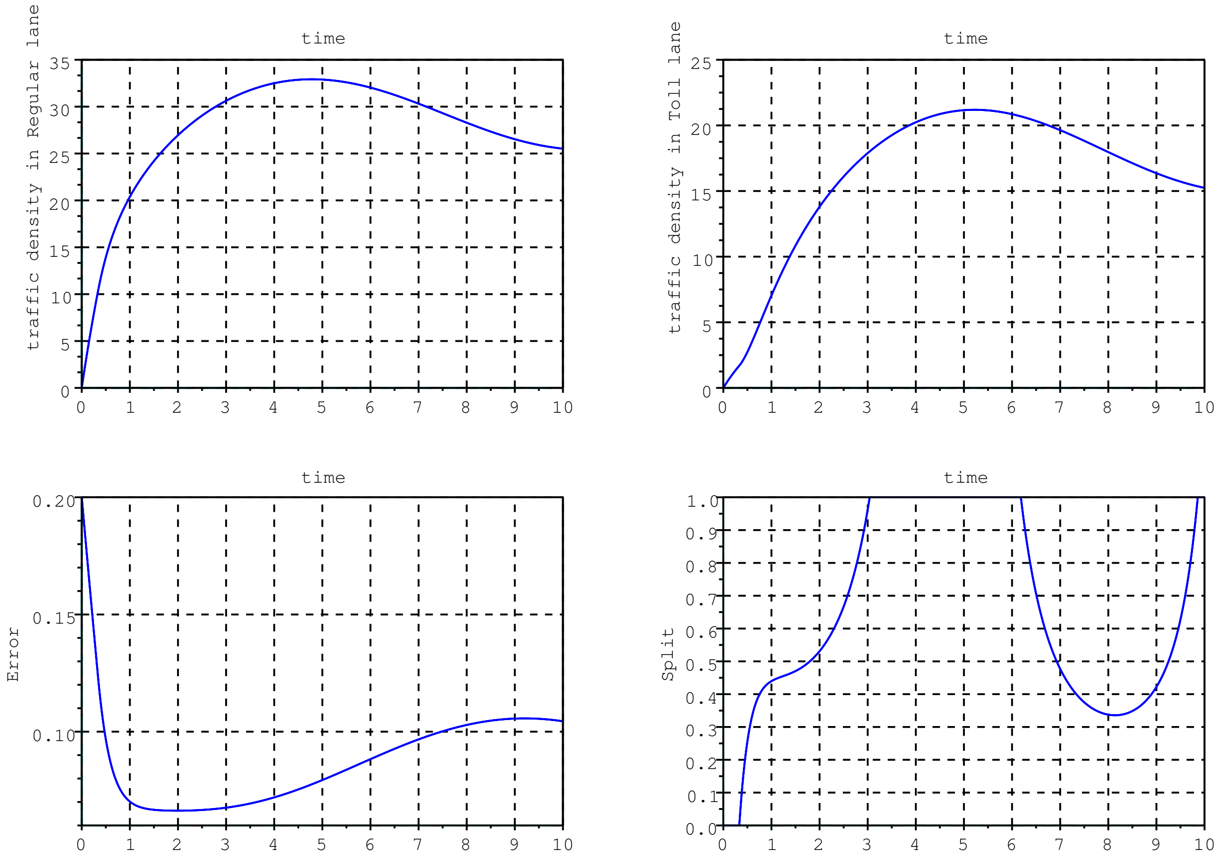

FIGURE 3 Feedback Tolling Results

TABLE 4 Simulation Parameters

SYMBOL	VALUE
β	0.75
S_T	6
ρ_m	120
v_f	1
γ	1.2
L_T	1
L_R	1

CONCLUSIONS

In this paper we formulated the mathematical models for different scenarios of tolling. The models allowed the flexibility for having RF tags based lanes and also regular lanes. The paper showed how to build models in a modular fashion to include the needed features. The models were then used to design real-time feedback controller using feedback linearization technique to regulate the traffic in the different lanes (or routes). Simulation software was developed using Scilab to show the robustness and performance of the algorithm, and it provided the validation for the control design.

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